ON THE THEORY OF VIBRATIONS OF ELASTIC BODIES WITH LIQUID CAVITIES

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One encounters various problems in modern technology that require the study of simultaneous vibrations of an elastic body and a liquid. The study of this problem in its general statement is complex. This paper presents an approximate theory that is based on the following simplifying assumptions: (a) Linearity of the problem: all displacements and velocities are assumed to be infinitesimally small, and consequently the equations of motion and the boundary conditions are linearized; (b) beam model: the real elastic body is replaced by a beam with a straight neutral axis, and the correctness of the hypothesis of plane sections is assumed; (c) the fluid is ideal and incompressible, and its motion is irrotational; (d) the body force is the force of gravity; (e) the external forces are conservative. Presented here are: the derivation of the general equations, the solvability of fundamental problems, an analysis of the spectrum, a formulation of the variational principles and their derivation.

1. Plane flexural vibrations of a beam with a cavity completely filled with liquid. 1. Introduce a coordinate system in the following manner (Fig. 1): The y-axis is directed along the neutral axis, the x- and z-axes are fixed in a section perpendicular to the y-axis so that the coordinate system is a right-handed system. The length of the beam will be denoted by l, the mass per unit length by $m(y)$, the bending rigidity by $c(y)$. We shall study only the case in which the vibrations take place in the yz-plane. We shall denote the deflection by $z(y, t)$.

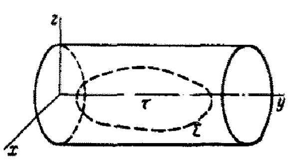

Fig. 1.

Let τ be the volume occupied by the fluid, Σ the surface bounding it, $\phi(x, y, z, t)$ the velocity potential of the absolute motion of the liquid. ϕ is a harmonic function in τ. On Σ it satisfied the condition

$$\partial\varphi / \partial n = v_n \tag{1.1}$$

where v_n is the projection of the velocities of points of Σ upon the exterior normal to Σ. In the given case

$$v_n = \frac{\partial z}{\partial t}(\mathbf{z}^0 \cdot \mathbf{n}^0) = \frac{\partial z}{\partial t}\gamma(x, y, z)$$

Here $\mathbf{n}^0$ is the unit vector of the exterior normal, $\gamma = \cos(\mathbf{z}^0\mathbf{n}^0)$.

Let $H(P, Q)$ be the Green function of the Neumann problem for the region τ. Let us introduce the Neumann operator

$$\mathrm{H}\,u = \int_{\Sigma} H(P, Q)\, u(Q)\, ds_Q, \qquad \varphi = \mathrm{H}\frac{\partial Z}{\partial t}\gamma \tag{1.2}$$

Construct expressions for the kinetic and potential energy of a beam with a liquid

$$T = \frac{1}{2}\int_0^l m Z_t^2\, dy + \frac{1}{2}\rho\int_{\tau}(\nabla\, H Z_t\gamma)^2\, d\tau, \qquad \Pi = \frac{1}{2}\int_0^l c Z_{yy}^2\, dy + \frac{1}{2}\int_0^l \beta Z^2\, dy \tag{1.3}$$

where ρ is the density of the liquid. The first part in the last equation is the potential energy of the elastic forces, the second part is that of the external forces, which are assumed to be conservative.

2. We set the problem of finding the free vibrations of the beam. For this we let first

$$Z(y; t) = \cos\omega t\,\vartheta(y), \qquad \varphi = -\omega\sin\omega t\,\Phi(x, y, z), \qquad \Phi = H\vartheta\gamma \tag{1.4}$$

Since the use of the Ritz method is anticipated, we write down the equation of Hamilton's principle

$$\delta L = \delta\int_0^t (T - \Pi)\, dt = 0 \tag{1.5}$$

After substituting in this equation the expressions for the function Z and ϕ from (1.4) and also putting $t = 2\pi/\omega$, and by discarding an insignificant multiplier, we reduce the expression for L to

$$L = \omega^2\left\{\int_0^l m\vartheta^2\, dy + \rho\int_{\tau}(\nabla\, H\vartheta\gamma)^2\, d\tau\right\} - \int_0^l c\vartheta''^2\, dy \int_0^l \beta\vartheta^2 dy \tag{1.6}$$

Now let $\{\psi_n\}$ be some complete set of functions orthonormalized in $[0, l]$. Then according to the Ritz method we must assume

$$\vartheta = \sum a_n\psi_n \tag{1.7}$$

where a_n are numbers to be determined. By substitution of series (1.7) into the expression for the functional L we obtain

$$L = \omega^2 \sum_{n,m} a_n a_m (\alpha_{nm} + \gamma_{nm}) - \sum_{n,m} a_n a_m (c_{nm} + \beta_{nm})$$

where

$$\alpha_{nm} = \int_0^l m\psi_n\psi_m\, dy, \qquad c_{nm} = \int_0^l c\psi_n''\psi_m''\, dy$$

$$\gamma_{nm} = \rho \int_\tau \nabla H\psi_n\gamma \cdot \nabla H\psi_m\gamma\, d\tau, \qquad \beta_{nm} = \int_0^l \beta\psi_n\psi_m\, dy \tag{1.8}$$

Thus, equation (1.5) will be reduced to a system of algebraic equations

$$\frac{\partial L}{\partial a_n} = \sum_m a_m \{\omega^2 (\alpha_{nm} + \gamma_{nm}) - (c_{nm} + \beta_{nm})\} \qquad (n = 1, 2, \ldots) \tag{1.9}$$

The characteristic equation of this system

$$|\, \omega^2 (\alpha_{nm} + \gamma_{nm}) - (c_{nm} + \beta_{nm})\, | = 0 \tag{1.10}$$

will be the frequency equation.

The coefficients γ_{nm} can be called coefficients of additional fluid mass, corresponding to the set of functions ψ_n. Thus the inertia properties of the fluid are determined by a symmetric matrix of infinite order. From the fact that in the general case $\gamma_{nm} \neq 0$ if $n \neq m$, it follows that the presence of the fluid inside the cavity changes not only the natural frequencies, but also the fundamental modes of vibration.

3. The foregoing section contained an exposition of the formal scheme of applying the Ritz method, and it was established that the problem of hydromechanics can be solved independently of the dynamical problem of the system. (The functions $H\psi_n\gamma$ depend only on the geometry of the cavity and the choice of the ψ_n functions, and do not depend on the motion of the beam).

When the computation scheme outlined above is realized, one encounters two questions. First, how to determine rationally the set of functions ψ_n, and second, how to construct the functions $H\psi_n\gamma$ effectively.

Concerning the second question, it is very difficult to give any general recommendations. With regard to the first question, one can in many cases recommend choosing for the functions $\{\psi_n\}$ the characteristic functions of the operator

$$\mathrm{L}\, u \equiv [cu_{yy}]_{yy} + \beta u = \lambda^2 mu \tag{1.11}$$

which satisfy boundary conditions corresponding to the conditions of beam support. If, for instance, both ends of the beam are free, then

$$u''(l) = u''(0) = 0, \quad u'''(l) = u'''(0) = 0 \tag{1.12}$$

One can easily verify that in the case of boundary conditions (1.12) the operator L is self-conjugate and that the characteristic functions of equation (1.11) posses the following properties: they are orthonormal with the weight $m(y)$

$$\int_0^l m(y)\, u_n(y)\, u_m(y)\, dy = \begin{cases} 0 & (m \neq n) \\ 1 & (m = n) \end{cases} \tag{1.13}$$

and also

$$\int_0^l c(y)\, u_n''(y)\, u_m''(y)\, dy + \int_0^l \beta u_n u_m\, dy = \begin{cases} 0 & (m \neq n) \\ \lambda_n & (m = 0) \end{cases} \tag{1.14}$$

where λ_n is the nth characteristic value of the L operator.

These properties permit a great simplification of the system (1.9), since in this case

$$\beta_{nm} + c_{nm} = \delta_{nm}\lambda_n^2, \qquad \alpha_{nm} = \delta_{nm} \qquad (\delta_{nm} \text{ is the Kronecker delta}) \tag{1.15}$$

The characteristic equation (1.10) then becomes

$$\begin{vmatrix} \omega^2(1+\gamma_{11}) - \lambda_1^2 & \omega^2\gamma_{12} & \omega^2\gamma_{13} & \ldots \\ \omega^2\gamma_{21} & \omega^2(1+\gamma_{22}) - \lambda_2^2 & \omega^2\gamma_{23} & \ldots \\ \omega^2\gamma_{31} & \omega^2\gamma_{32} & \omega^2(1+\gamma_{33}) - \lambda_3^2 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0 \tag{1.16}$$

The infinite system (1.9) and the determinant (1.10) or (1.16) are always convergent. This fact follows from the general theory (see Section 3).

If the beam is homogeneous and the external forces are uniformly distributed along the span (the functions $c(y)$, $\beta(y)$ and $m(y)$ are constants) then the construction of the set $\{\psi_n\}$ is elementary. If the parameters of the beam vary along its span, then the construction of the system of coordinate functions is considerably more complicated.

4. Zhukovskii showed that from the dynamic point of view the rigid body which contains a cavity that is completely (without a free surface) filled with an ideal incompressible fluid is similar to some rigid body without the fluid. The mass of such an "equivalent" rigid body is equal to the sum of the masses of the rigid body and the fluid, and the inertia

tensor is determined by the density of the fluid and the geometry of the cavity.

It is interesting to explore to what degree this fact influences the phenomenon under investigation, i.e. can one maintain that from the dynamic point of view the beam with a fluid is equivalent to some beam without fluid but with some other mass distribution along the cross-sections?

In order to settle this question we set up the differential equations of motion for the system under study.

Let us compose the Hamiltonian equation

$$L = L_1 + L_2$$

where

$$L_1 = \frac{1}{2}\int_0^t\int_0^l \{mZ_t^2 - cZ_{yy}^2 - \beta Z^2\}\, dy\, dt, \qquad L_2 = \frac{1}{2}\rho\int_0^t\int_\tau (\nabla \mathrm{H}\,\gamma Z_t)\, d\tau\, dt$$

The variation of the first functional is computed by the standard method. It is equal to

$$\delta L_1 = -\int_0^t\int_0^l \{mZ_{tt} + (cZ_{yy})_{yy} + \beta Z\}\,\delta Z\, dy\, dt \tag{1.17}$$

We compute the variation of the second functional

$$\delta L_2 = \rho\int_0^t\int_\tau (\nabla \mathrm{H}\,\gamma Z_t)\cdot(\nabla \mathrm{H}\,\gamma\delta Z_t)\, d\tau\, dt$$

then apply Green's formula

$$\delta L_2 = \rho\int_0^t\int_\Sigma \mathrm{H}\,\gamma Z_t \frac{\partial}{\partial n_\Sigma} \nabla \mathrm{H}\,\gamma\delta Z_t\, d\Sigma\, dt$$

Since, after determining the H operator, $\partial \mathrm{H}u/\partial n_\Sigma = u$, the expression for L_2 can be simplified as follows:

$$\delta L_2 = \rho\int_{y_1}^{y_2} dy\int_0^t \frac{\partial}{\partial t}\,\delta Z F(y, t)\, dt \qquad \left(F(y, t) = \int_{l_y} (\mathrm{H}\,\gamma Z_t)\,\gamma\, dl\right) \tag{1.18}$$

Here l_y denotes the perimeter of the cross-section normal to the y-axis, whose ordinate equals y. Integrating (1.18) by parts with respect to t and taking into account the convergence of the variation yields

$$\delta L_2 = -\rho\int_0^t\int_{y_1}^{y}\int_{l_y} (\mathrm{H}\,\gamma Z_{tt})\,\gamma\, dl\,\delta Z\, dy\, dt \tag{1.19}$$

The integration with respect to y in this expression can be assumed to be performed within the limits zero to l. For that purpose it is sufficient to let $H \equiv 0$ for values of y lying outside the interval $[y_1, y_2]$.

According to Hamilton's principle

$$\delta L_1 + \delta L_2 = 0$$

When one substitutes into this equation the expression for the variations of L_1 and L_2 and uses the differential of the quantity δZ, one arrives at the following integro-differential equation for flexural vibrations of the beam which contains a fluid inside:

$$\rho \int_{l_y} (H\gamma Z_{tt}) \gamma \, dl + mZ_{tt} + (cZ_{yy})_{yy} + \beta Z = 0 \qquad (1.20)$$

The first integral takes into account the inertia of the fluid. If it is equal to zero then we have the equation of flexural vibrations of a beam without a liquid. That is well explored. In this case the accelerations at any cross-section are determined uniquely by the values of the elastic force and the external force at that cross-section. The value of the first term is determined by the character of the accelerations of all cross-sections of the beam. In other words, the hypothesis of plane sections is known to be incorrect for the fluid (fluid particles displace along the y-axis). Consequently, it is impossible to introduce an equivalent beam in the general case.

5. The simplifications which are introduced into the calculations by reducing the problem of the beam with a liquid to a problem of an equivalent ordinary beam are so significant that it is natural to explore those conditions under which such a replacement does not lead to large errors. A detailed exposition of this problem requires a rather large amount of space. Therefore, we shall present here only the final result.

The hypothesis of plane sections can be used only in the case where the length of the beam is large compared to its other dimensions. In addition, it is necessary that the surface of the cavity differs only slightly from a cylindrical surface whose generator is parallel to the axis of the beam (y-axis). However, even in that case the hypothesis of plane sections can give a more or less accurate result only for the analysis of the first natural modes.

Let us assume now that the hypothesis of plane sections is correct. This means that the fluid moves only in the plane of the cross-sections, normal to the axis of the beam. Let us denote by $\phi_y(x, z)$ the velocity potential of the fluid flow at a cross-section, whose ordinate equals y and by H_y the Neumann operator for this section (a plane figure bounded by the contour l_y). Then

$$\varphi_y = H_y \gamma_y Z_t$$

Here $\gamma_y = \cos(\mathbf{n}_l^0 \mathbf{z}^0)$ is the direction cosine of the normal to l_y in the xZ-plane.

Introduce the notion of an additional fluid mass per unit length

$$m_{ж}(y) = \rho \int_{l_y} \gamma_y H_y \gamma_y \, dl$$

Then the kinetic energy of the beam-fluid system can be written in the following form:

$$T = \frac{1}{2} \int_0^l \{m(y) + m_{ж}(y)\} Z_t^2 \, dy \qquad (1.21)$$

This expression shows that in the case studied the problem reduces to the investigation of flexural vibrations of an "equivalent beam" having the same bending rigidity but a changed mass per unit length.

$$m^* = m + m_f$$

2. Flexural vibrations of a beam in the case when the liquid substance contained inside has a free surface. 1. It was established in the previous section that if the fluid fills the cavity completely, then it does not add any new degrees of freedom. The motion of the fluid leads in this case only to some change in the magnitudes of the natural frequencies and principal mode shapes in comparison with those which the beam would have if it did not contain any liquid.

If the liquid has a free surface on which waves can be formed, then the motion of the liquid leads to the appearance of an additional natural frequency spectrum and additional principal modes.

Thus, let the fluid have a free surface. We introduce still another "fixed" coordinate system $x_1 y_1 z_1$ such that the plane $x_1 y_1$ coincides with the free surface of the fluid in the equilibrium position. The axis oz_1 will be directed upward (in the general case, in the direction of the body forces), as is shown in Fig. 2.

The region τ will now denote the volume bounded by the wetted surface of the cavity (surface Σ) and the free surface and the free surface (plane surface S). Because of the linearity of the problem the velocity potential of the absolute motion of the fluid can be written as follows:

$$\varphi(P, t) = \int_{\Sigma+S} H(P, Q)\, v_n(Q, t)\, ds_Q + \int_S H(P, Q)\left(\frac{\partial \zeta}{\partial t} - v_n\right) ds_Q =$$

$$= \mathrm{H}\, v_n + \mathrm{H}^*\left(\frac{\partial \zeta}{\partial t} - v_n\right) \qquad (2.1)$$

where $z_1 = \zeta(t, x_1 y_1)$ is the equation of the free surface and the velocity v_n is determined from the formula $v_n = Z_t \cos(nz) = Z_t \gamma$.

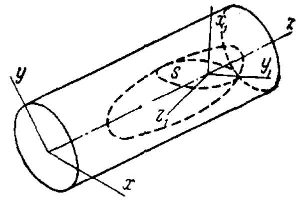

Fig. 2.

The operator H has been already determined. The operatory H* determines some function in τ whose normal derivative on Σ is zero, and on S is equal to $\partial\zeta/\partial t - v_n$.

Using the assumption of incompressibility and the hypothesis of plane sections (from which it follows that the volume of the beam remains unchanged) we establish that

$$\int_S \left(\frac{\partial \zeta}{\partial t} - v_n\right) ds = 0$$

From this it follows that the function $H^*(\partial\zeta/\partial t - v_n)$ is harmonic in τ.

To proceed further it is convenient to introduce the quantity

$$U = \zeta - Z\gamma = \zeta - z_1$$

U is the additional displacement of the liquid due to the waves forming on its surface, z_1 is the displacement of the points on the surface S due to bending computed in terms of the $x_1 y_1 z_1$ coordinates.

The kinetic energy of the system investigated is given by

$$T = \frac{1}{2}\int_0^l m Z_t^2\, dy + \frac{1}{2}\rho \int_\tau (\nabla \mathrm{H}\gamma Z_t)^2 d\tau +$$
$$+ \rho \int_\tau (\nabla \mathrm{H}\,\gamma Z_t)(\nabla \mathrm{H}^* U_t)\, d\tau + \frac{1}{2}\rho \int_\tau (\nabla \mathrm{H}^* U_t)^2\, d\tau \qquad (2.2)$$

The potential energy of this system is equal to the sum of the potential energies of the elastic and external forces and the potential energy of the fluid (which is determined from the change of the position of the center of gravity of the liquid mass):

$$\Pi = \frac{1}{2}\int_0^l c\,(Z_{yy})^2\, dy + \frac{1}{2}\int_0^l \beta Z^2\, dy + \rho g \int_\tau z_1\, d\tau$$

Let us evaluate the last integral in this sum:

$$\int z_1\, d\tau = \frac{1}{2}\int_S \zeta^2\, dx_1\, dy_1 - \frac{1}{2}\int_S z_1^{*2}\, dx_1\, dy_1$$

Here $z_1 = z_1^*(x_1, y_1, t_1)$ is the equation of the Σ surface in the $x_1 y_1 z_1$ coordinate system. Since $\zeta = U + z_1$

$$\int_{\tau} z_1\, d\tau = \frac{1}{2}\int_S U^2\, dx_1\, dy_1 + \int_S UZ_{\gamma}\, dx_1\, dy_1 + J$$

The last part of this sum has the following structure:

$$J = \int_{y_1}^{y_2} Z^2(y, t)\, F(y)\, dy$$

where $F(y)$ is a known function (which does not depend on time). Thus, this part has the same structure as the potential energy of the external forces. It should be so because J determines that part of the energy of the force of gravity which depends only on the deflection. Thus, the value J can be included in the expression for the potential energy of the external forces after changing the function β in a suitable manner. Thus we will have

$$\Pi = \frac{1}{2}\int_0^l cZ_{yy}^2\, dy + \frac{1}{2}\int_0^l \beta Z^2\, dy + \frac{1}{2}\rho g\int_S U^2\, ds + \rho g\int_S UZ\gamma\, ds \qquad (2.3)$$

2. The equations of motion can be constructed in the simplest way if one uses Hamilton's principle

$$\delta L = 0, \qquad L = L_1 + L_2 \qquad (2.4)$$

where

$$L_1 = \frac{1}{2}\int_0^t\int_0^l mZ_t^2\, dy\, dt + \frac{1}{2}\rho\int_0^t\int_{\tau} (\nabla H\gamma Z_t)^2\, d\tau\, dt - \frac{1}{2}\int_0^t\int_0^l [cZ_{yy}^2 + \beta Z^2]\, dy$$

$$L_2 = \rho\int_0^t\int_{\tau} (\nabla H\gamma Z_t)(\nabla H^* U_t)\, d\tau\, dt + \frac{1}{2}\rho\int_0^t\int_{\tau} (\nabla H^* U_t)^2\, d\tau\, dt -$$

$$- \frac{1}{2}\rho g\int_0^t\int_S U^2\, ds\, dt - \rho g\int_0^t\int_S UZ\gamma\, ds\, dt$$

The variation of the first functional was evaluated in the previous section of the paper:

$$\delta L_1 = -\int_0^t\int_0^l \left\{mZ_{tt} + (cZ_{yy})_{yy} + \beta Z + \rho\int_{l_y} \gamma H\, \gamma Z_{tt}\, dl\right\}\delta Z\, dy\, dt \qquad (2.5)$$

Let us evaluate the variation of the second functional:

$$\delta L_2 = \rho\int_0^t\int_{\tau} (\nabla H\, \gamma Z_t)(\nabla H^*\, \delta U_t)\, d\tau\, dt + \rho\int_0^t\int_{\tau} (\nabla H\, \gamma\delta Z_t)(\nabla H^*\, U_t)\, dt\, d\tau +$$

$$+ \rho\int_0^t\int_{\tau} \nabla H^*\, U_t\cdot\nabla H^*\, \delta U_t\, d\tau\, dt -$$

$$-\rho g\int_0^t\int_S U\delta U\,ds\,dt-\rho g\int_0^t\int_S U\gamma\delta Z\,ds\,dt-\rho g\int_0^t\int_S \gamma Z\,\delta U\,ds\,dt \tag{2.6}$$

Use Green's formula and the fact that

$$\frac{\partial Hv}{\partial n}=v \quad (P\in\Sigma+S), \qquad \frac{\partial H^*v}{\partial n}=\begin{cases} v & (P\in S)\\ 0 & (P\in\Sigma)\end{cases}$$

This allows us to represent expression (2.6) in the following form

$$\delta L_2=\rho\int_0^t\int_{\Sigma+S}\gamma H^*U_t\,\delta Z_t\,ds\,dt+\rho\int_0^t\int_S H\gamma Z_t\,\delta_t U\,ds\,dt+\rho\int_0^t\int_S \mathrm{H}^*U_t\,\delta U_t\,ds\,dt-$$
$$-\rho g\int_0^t\int_S U\delta U\,ds\,dt-\rho g\int_0^t\int_S\gamma U\,\delta Z\,ds\,dt-\rho g\int_0^t\int_S\gamma Z\,\delta U\,ds\,dt$$

Integration by parts (with respect to t) and utilization of the isochronous property of the variation yields

$$\delta L_2=-\rho\int_0^t\int_S\{\mathrm{H}\gamma Z_{tt}+H^*U_{tt}+gU+g\gamma Z\}\,\delta U\,ds\,dt-$$
$$-\rho\int_0^t\int_{\Sigma+S}\gamma\mathrm{H}^*U_{tt}\,\delta Z\,ds\,dt-\rho g\int_0^t\int_S\gamma U\,\delta Z\,ds\,dt \tag{2.7}$$

By substituting (2.5) and (2.7) into equation (2.4) and using the independence of the variations we obtain the following equation:

$$mZ_{tt}+\rho\int_{l_y}\gamma\mathrm{H}\,\gamma Z_{tt}\,dl+\rho\int_{l_y}\gamma\mathrm{H}^*U_{tt}\,dl+(cZ_{yy})_{yy}+\beta Z+\rho g\int_d\gamma U\,dl=0$$
$$\mathrm{H}\gamma Z_{tt}+\mathrm{H}^*U_{tt}+\rho gU+\rho g\,\gamma Z=0 \tag{2.8}$$

Here d denotes the line of intersection of the surface S with the plane of a normal cross-section.

Thus the motion of the beam-fluid system in the given case is described by a system of two integro-differential equations.

3. We shall use the Ritz method to determine the natural frequencies and principal mode shapes. To do it we substitute into equation (2.4)

$$Z(y,t)=\cos\omega t\,\vartheta(y)\quad y\ \text{on}\ [0;\,l],\qquad U(P,t)=\cos\omega t\,f(P)\quad P\ \text{on}\ S \tag{2.9}$$

The problem of determining the natural vibrations reduces to the minimizing the following functional:

$$L^* = \omega^2 \left\{ \frac{1}{2} \int_0^l m\vartheta^2 dy + \frac{1}{2}\rho \int_\tau (\nabla H\gamma\vartheta)^2 \, d\tau + \rho \int_\tau (\nabla H\gamma\vartheta)(\nabla H^* f)\, d\tau + \right.$$
$$\left. + \frac{1}{2}\rho \int_\tau (\nabla H^* f)^2 \, d\tau \right\} - \frac{1}{2} \int_0^l [(c\vartheta'')^2 + \beta\vartheta^2]\, dy - \frac{1}{2} g\rho \int f^2 \, ds - \rho g \int_S \gamma f\vartheta ds \quad (2.10)$$

In order to solve this variational problem by the Ritz method it is necessary to choose two sets of coordinate functions $\psi_n(y)$ and $\chi_n(P)$, which are complete and orthonormal on $[0, l]$ and S, respectively. Then we put

$$\zeta = \Sigma a_n \psi_n, \qquad f = \Sigma b_n \chi_n$$

After that this problem is reduced in the usual manner to a system of homogeneous algebraic equations with a symmetric matrix.

Thus the problem of the study of free vibrations of a beam inside which there is a liquid having a free surface reduces to constructing the H and H* operators (for this purpose it is necessary to be able to solve numerically the Neumann problem for the region τ), to constructing a set of functions ψ_n and χ_n, and to setting up and solving a homogeneous system of algebraic equations.

The previous sections contained a formal analysis of the problem. It was demonstrated by means of an example of a simple "one-dimensional" motion that it is possible to reduce the problem to an algebraic one. The general case of small vibrations of a beam is studied below, where the main attention is focused on the problem of the existence of principal modes and the properties of the spectrum.

3. Arbitrary vibrations of the beam. Some general problems of the theory. *1. Reduction to a functional equation and some simplifications.* Let us denote the deflection in the xOy plane by $X(y, t)$ and the angle of twist by $\theta(y, t)$. The kinetic and potential energy of the beam without the fluid can be written as

$$T_1 = \frac{1}{2} \int_0^l \left\{ A_{11}X_t^2 + 2A_{12}X_tZ_t + 2A_{13}X_t\theta_t + A_{22}Z_t^2 + 2A_{23}Z_t\theta_t + A_{33}\theta_t^2 \right\} dy$$
$$\Pi_1 = \frac{1}{2} \int_0^l \{ C_1X_{yy}^2 + C_2Z_{yy}^2 + C_3\theta_y^2 + B_1X^2 + B_2Z^2 + B_3\theta^2 \}\, dy \quad (3.1)$$

Here C_i are the bending and torsion rigidities and B_i are functions characterizing the external forces.

The normal velocity component of points on the surface of the cavity is expressed in our case by the formula

$$v_n = X_t \gamma_1 + Z_t \gamma_2 + \theta_t \gamma_3$$

where

$$\gamma_1 = \cos(n\mathbf{x}^\circ), \quad \gamma_2 = \cos(n\mathbf{z}^\circ) \quad \gamma_3 = z\cos(n\mathbf{x}^\circ) - x\cos(n\mathbf{z}^\circ)$$

Introduce the elastic displacement of points in the plane:

$$d_n = X\gamma_1 + Z\gamma_2 + \theta\gamma_3$$

Similarly, as it was done in the previous section, we shall introduce a function $U(P, t)$:

$$U(P, t) = \zeta(P, t) - d_n(P, t) \qquad (P \in S)$$

Repeating then the reasoning of the previous sections we can write the expressions for the energies, the functional L and the integro-differential equation of motion

$$T = T_1 + \tfrac{1}{2}\rho\int_\tau \{\nabla \mathrm{H} X_t\gamma_1 + \nabla \mathrm{H} Z_t\gamma_2 + \nabla \mathrm{H}\theta_t\gamma_3 + \nabla \mathrm{H}^* U_t\}^2 d\tau$$
$$\Pi = \Pi_1 + \tfrac{1}{2}\rho g\int_S U^2\, ds + \rho g\int_S U\{X\gamma_1 + Z\gamma_2 + \theta\gamma_3\}\, ds \tag{3.2}$$

$$L = \int_0^t (T - \Pi)\, dt \tag{3.3}$$

$$A_{11}X_{tt} - \rho\int_{l_y} \gamma_1 \mathrm{H} X_{tt}\gamma_1\, dl + A_{12}Z_{tt} + \rho\int_{l_y}\gamma_1\mathrm{H}Z_{tt}\gamma_2\, dl + A_{13}\theta_{tt} + \tag{3.4}$$
$$+ \rho\int_{l_y}\gamma_1\mathrm{H}\theta_{tt}\gamma_3\, dl + \rho\int_{l_y}\gamma_1\mathrm{H}^*U_{tt}\, dl + (C_1X_{yy})_{yy} + \beta_1 X + \rho g\int_d \gamma_1 U\, dl = 0$$

$$A_{12}X_{tt} + \rho\int_{l_y}\gamma_2\mathrm{H}X_{tt}\gamma_1\, dl + A_{22}Z_{tt} + \rho\int_{l_y}\gamma_2\mathrm{H}Z_{tt}\gamma_2\, dl + A_{23}\theta_{tt} + \tag{3.5}$$
$$+ \rho\int_{l_y}\gamma_2\mathrm{H}\theta_{tt}\gamma_3\, dl + \rho\int_{l_y}\gamma_2\mathrm{H}^*U_{tt}\, dl + (C_2Z_{yy})_{yy} + \beta_2 Z + \rho g\int_d\gamma_2 U\, dl = 0$$

$$A_{13}X_{tt} + \rho\int_{l_y}\gamma_3\mathrm{H}X_{tt}\gamma_1\, dl + A_{23}Z_{tt} + \rho\int_{l_y}\gamma_3\mathrm{H}Z_{tt}\gamma_2\, dl + A_{33}\theta_{tt} + \tag{3.6}$$
$$+ \rho\int_{l_y}\gamma_3\mathrm{H}\theta_{tt}\gamma_3\, dl + \rho\int_{l_y}\gamma_3\mathrm{H}^*U_{tt}\, dl - (C_3\theta_y)_y + \beta_3\theta + \rho g\int_d\gamma_3 U\, dl = 0$$

$$\mathrm{H}X_{tt}\gamma_1 + \mathrm{H}Z_{tt}\gamma_2 + \mathrm{H}\theta_{tt}\gamma_3 + \mathrm{H}^*U_{tt} + \rho g\gamma_1 X + \rho g\gamma_2 Z + \rho g\gamma_3\theta + \rho g U = 0 \tag{3.7}$$

In order to study the existence of periodic solutions of the system (3.4) to (3.7) and to analyse the properties of the spectrum, it is expedient to write this system in operational form.

Let us introduce the following function spaces:

(a) Function spaces E_1 and E_2 of functions $u_1(y)$ and $u_2(y)$ with a summed square on $[0\ l]$, possessing generalized fourth derivatives and a scalar product

$$(u_i, v_i)_i = \int_0^l u_i v_i \, dy \qquad (i = 1, 2) \tag{3.8}$$

(b) A function space E_3 of functions $u_3(y)$ with a summed square on $[0\ l]$, possessing generalized second derivatives and a product of the type (3.8).

(c) A Hilbert space E_4 of functions $u_4(P), P \in S$ with a summed square on S and with a scalar product of the form $E = E_1 + E_2 + E_3 + E_4$

$$(u_4, v_4)_4 = \int_S u_4 v_4 \, dx_1 \, dy_1 \tag{3.9}$$

(d) A direct sum of the function spaces $E = E_1 + E_2 + E_3 + E_4$ with a scalar product

$$(x, y) = \sum_{i=1}^{4} (u_i v_i)$$

Here x is a vector with components u_1, u_2, u_3, u_4.

The functions u_i should also satisfy some boundary conditions which are determined from the type of beam support. We shall not specify these conditions in detail. We shall assume them to be homogeneous and such that they insure the self-conjugate property of the operators which describe the elastic vibrations of the beam without the fluid.

We introduce the operators L_{ij} and M_{ij} acting from E_j to E_i:

$$\mathrm{L}_{ij} u_j = A_{ij} u_j + \rho \int_{l_y} \gamma_i \mathrm{H} \gamma_j u_j \, dl \quad (i, j = 1, 3, 2), \qquad \mathrm{L}_{4j} u_j = \mathrm{H} \gamma_j u_j \quad (i = 1, 2, 3)$$

$$\mathrm{L}_{i4} u_4 = \rho \int_{l_y} \gamma_i \mathrm{H}^* u_4 \, dl \quad (i = 1, 2, 3), \qquad \mathrm{L}_{44} u_4 = \mathrm{H}^* u_4$$

$$\mathrm{M}_{ii} u_i = (c_i u_{iyy})_{yy} + \beta_i u_i \qquad (i = 1, 2), \quad \mathrm{M}_{j4} u_4 = \rho g \int_d \gamma_i u_4 \, dl \quad (j = 1, 2, 3)$$

$$\mathrm{M}_{33} u_3 = -(c_3 u_{3y})_y + \beta_3 u_3, \qquad \mathrm{M}_{44} u_4 = \rho g u_4, \quad \mathrm{M}_{4j} u_j = \rho g \gamma_j u_j \quad (j = 1, 2, 3)$$

Since H is an integral operator (1.2), whose kernel is a Green's function, then in order to have the operators L_{ij} and M_{ij} act from L_j to L_i, it is necessary to impose some limitations upon the functions A_{ij}, C_i and B_i. We shall not stop to consider these questions, but assume once and for all that these functions satisfy all necessary conditions.

Let us also introduce operators L and M in terms of the following equalities:

$$\mathrm{L} = \begin{pmatrix} \mathrm{L}_{11} \ldots \mathrm{L}_{14} \\ \ldots\ldots \\ \mathrm{L}_{41} \ldots \mathrm{L}_{44} \end{pmatrix}, \qquad \mathrm{M} = \begin{pmatrix} \mathrm{M}_{11} & 0 & 0 & \mathrm{M}_{14} \\ 0 & \mathrm{M}_{22} & 0 & \mathrm{M}_{24} \\ 0 & 0 & \mathrm{M}_{33} & \mathrm{M}_{34} \\ \mathrm{M}_{41} & \mathrm{M}_{42} & \mathrm{M}_{43} & \mathrm{M}_{44} \end{pmatrix}$$

With this notation the system (3.4) to (3.7) becomes

$$\mathrm{L}x_{tt} + \mathrm{M}x = 0 \quad . \tag{3.10}$$

Let us explain some properties of the L and M operators. Operator L is self-conjugate, i.e.

$$(\mathrm{L}x, y) = (x, \mathrm{L}y) \quad \text{or} \quad \sum_{i,\, j=1}^{4} (\mathrm{L}_{ij} u_j, v_i)_i = \sum_{i,\, j=1} (u_i, \mathrm{L}_{ij} v_j)_i \tag{3.11}$$

The validity of equations of this kind is established by a simple verification.

Thus, the L operator is self-conjugate. However, it is not completely continuous. This circumstance complicates the proof.

Let us now proceed to the study of the M operator. Calculate $(\mathrm{M}x, x)$:

$$(\mathrm{M}x, x) = \int_0^l (c_1 u_{1yy})_{yy} u_1\, dy + 2\rho g \int_0^l u_1 \int_d \gamma_1 u_4\, dl\, dy + \int_0^l \beta_1 u_1^2\, dy +$$
$$+ \int_0^l (c_2 u_{2yy})_{yy} u_2\, dy + 2\rho g \int_0^l u_2 \int_d \gamma_2 u_4\, dl\, dy + \int_0^l \beta_2 u_2^2\, dy + \int_0^l (-c_3 u_{3y})_y u_3\, dy +$$
$$+ 2\rho g \int_0^l u_3 \int_d \gamma_3 u_3\, dl\, dy + \int_0^l \beta_3 u_3^2\, dy + \rho g \int_S u_4^2\, ds$$

After comparing this expression with equation (3.2) we have

$$(\mathrm{M}x \cdot x) = 2\Pi \tag{3.12}$$

Remark. In order to be convinced of the validity of formula (3.12), it is necessary to use the self-conjugate property of the boundary conditions for the functions $u_i (i = 1, 2, 3)$.

In order that the problem make sense it is necessary that the functional Π be positive definite. Questions regarding the conditions which have to be satisfied for that purpose by functions c_i, γ_i and β_i are not trivial. However, this investigation is greatly simplified because of the following fact.

Theorem. For the functional Π to be positive definite it is necessary and sufficient that the functional Π^* be positive definite, where

$$\Pi^* = \Pi_1 - \rho g \int_S \gamma_1 \gamma_2 ZX\, ds - \rho g \int_S \gamma_1 \gamma_3 X\theta\, ds - \rho g \int_S \gamma_2 \gamma_3 Z\theta\, ds - \tag{3.13}$$

$$-\frac{1}{2}\rho g\int \gamma_1{}^2X^2\,ds-\frac{1}{2}\rho g\int_S \gamma_2{}^2 Z^2\,ds-\frac{1}{2}\rho g\int_S \gamma_3{}^2\theta^2\,ds$$

and the functional Π_* is determined from formula (3.1).

In order to be convinced of the validity of this theorem, it is sufficient to perform the following substitution in the expression for the potential energy:

$$U=v-\gamma_1X-\gamma_2Z-\gamma_3\theta \tag{3.14}$$

After that the functional Π has the form

$$\Pi=\Pi^*+\frac{1}{2}\rho g\int_S v^2\,ds$$

This theorem, which is a generalization of the analogous theorem of the theory of motion of a rigid body with a fluid (see reference [2]), has a principal character. It shows that the question of the stability of a beam, inside which there is a liquid mass having a free surface, reduces to the study of the stability of the same beam without a fluid but under the action of another system of external forces. The potential energy of the changed system of forces differs from the potential energy of the original system by terms that are identically determined by the density of the fluid and the geometry of the cavity.

The positive definite quality of the functional Π^* will thus denote the fact that in the equilibrium state the potential energy of the beam-fluid system has a minimum, and consequently this state of equilibrium will be stable. (Strictly speaking, this statement is still to be proved, as will be done below).

In the following we shall assume that the functional Π^*, and consequently also the operator M, are positive definite.

The operator M is self-conjugate. This fact is established by a direct verification.

2. *Study of the quadratic functional* Π^*. It is rather complicated to study in general the necessary and sufficient conditions for positive definiteness of the functional Π^*. However, it is very simple to set up the sufficient conditions only, and in addition they will have a simple physical meaning.

Let us first study the simplest case of only torsional vibrations of the beam. The functional Π^* will then have the form

$$\Pi^*=\frac{1}{2}\int_0^l\{c_3\theta_y{}^2+\beta_3\theta^2-b_3\theta^2\}\,dy \qquad \left(b_3(y)=\rho g\int_d \gamma_3{}^2\,dl\right) \tag{3.15}$$

If one assumes that C_3 never goes to zero, then one of the sufficient conditions can be written immediately (see, for instance, [4], p. 122):

$$\beta_3 > b_3 \tag{3.16}$$

Using the expression $\gamma_3 = z \cos nx - x \cos nz$, equation (3.15) can be written in the form

$$b_3(y) = \rho g \{J_{zz}(y) \cos^2(nz) + J_{xx} \cos^2(nx) \mp 2J_{zx} \cos(nx) \cos(nz)\} \tag{3.17}$$

Here J_{xx}, J_{zz}, J_{xz} denote moments of inertia of a segment d.

If the axis of the beam is horizontal then $\mathbf{z}^0 = \mathbf{n}^0$, and then

$$b(y) = \rho g J_{xx} \tag{3.18}$$

If the axis of the beam is vertical then $\mathbf{y}^0 = \mathbf{n}^0$, and then $b(y) = 0$, i.e. the fluid does not influence the sign-definiteness of Π. Thus, for the functional to be positive definite (and consequently, also for stability of the equilibrium position) it is sufficient to satisfy the condition which is identically determined by the geometrical characteristics of the free surface and the density of the fluid.

If the beam undergoes only flexural vibrations in one of the planes then similar conditions are derived analogously. In that case

$$\Pi^* = \frac{1}{2}\int_0^l \{c_1 X_{yy}{}^2 + (\beta_1 - b_1) X^2\}\, dy \qquad \left(b_1(y) = \rho g \int_d \gamma_1{}^2\, dl\right)$$

Since $\gamma_1 = \cos nx$, therefore $b_1 = \rho\, gd\, \cos^2 nx$. For future use let

$$J_{ik} = \rho g \int_d \gamma_i \gamma_k\, dl$$

The criterion for the form of (3.16) can also be easily obtained for the general case. For that purpose let us represent the functional Π^* in the form of a sum:

$$\Pi^* = \Pi_1^* + \Pi_2^*$$

where

$$\Pi_1^* = \frac{1}{2}\int_0^l \{c_1 X_{yy}^2 + c_2 Z_{yy}^2 + c_2 \theta_y{}^2\}\, dy$$

$$\Pi_2^* = \frac{1}{2}\int_0^l \{\beta_1 - J_{11}\} X^2\, dl + \frac{1}{2}\int_0^l \{\beta_2 - J_{22}\} Z^2\, dl + \frac{1}{2}\int_0^l \{\beta_3 - J_{33}\} \theta^2\, dl -$$
$$- \rho g \int_0^l\int_d \gamma_1\gamma_2 XZ\, dl\, dy - \rho g \int_0^l\int_d \gamma_1\gamma_3 X\theta\, dl\, dy - \rho g \int_0^l\int_d \gamma_2\gamma_3 Z\theta\, dl\, dy$$

For the functional Π^*, to be positive definite it is sufficient that for an arbitrary y $[0, l]$ the functions c_i satisfy the inequalities

$c_i > \delta_i$, where δ_i are arbitrary positive numbers that can be as small as is desired. For the functional $\Pi_2{}^*$ to be positive definite it is sufficient, for instance, that the following conditions be satisfied:

$$\beta_1 - J_{11} > \alpha_1, \qquad (\beta_1 - J_{11})(\beta_2 - J_{22}) - \rho^2 g^2 J_{12} > \alpha_2 \tag{3.19}$$

$$\begin{vmatrix} \beta_1 - J_{11} & -J_{12} & -J_{13} \\ -J_{21} & \beta_2 - J_{22} & -J_{23} \\ -J_{31} & -J_{32} & \beta_3 - J_{33} \end{vmatrix} > \alpha_3 \tag{3.20}$$

where α_3 are arbitrary positive numbers that can be as small as is desired.

These conditions are analogous to the Sylvester inequalities. They impose limitations upon the geometrical characteristics, and one can express them in graphical form. We shall demonstrate this with an example of flexural vibrations in two mutually perpendicular planes.

In this case conditions (3.20) can be written in the following form [1]

$$\beta_1 > \beta_1{}^* + \alpha_1, \qquad \beta_2 > \beta_2{}^* + \alpha_2, \qquad (\beta_1 - \beta_1{}^*)(\beta_2 - \beta_2{}^*) - (\beta + \alpha_3) > 0$$

Here

$$\beta_1{}^* = \rho g d \cos nx, \qquad \beta_2{}^* = \rho g d \cos nz, \qquad \beta = \rho^2 g^2 \cos nx \cos nz$$

Consequently, in order to satisfy these conditions, the quantities β_1 and β_2 should be such that for an arbitrary y $[0\,l]$ the point $\beta_1\beta_2$ lie in the shaded region of the plane $\beta_1 0\,\beta_2$ (Fig. 3).

The conditions that we have just considered are very crude. Intuitively, it is quite clear that if the rigidity is positive then in order to insure positive definiteness of Π^* it is not at all necessary that the external forces have a restoring character.

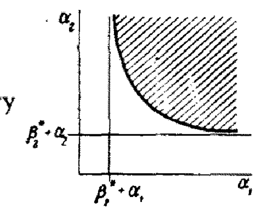

Fig. 3.

Let us study vibrations with "one degree of freedom", for instance, torsional vibrations. We have

$$\Pi^* = \int_0^l c_3 \theta_y{}^2 dy + \int_0^l r(y)\,\theta^2 dy \qquad \left(r(y) = \beta_3 - \rho g \int_d \gamma^2 dl\right)$$

and let $r \geqslant -\delta$ for y on $[0\,l]$, where δ is some positive number. Then

$$\Pi^* \geqslant \int_0^l c_3 \theta_y{}^2 dy - \delta \|\theta\|^2$$

Evaluate from below the first term of the right-hand side. Since

$$\theta - \theta_0 = \int_0^y \theta_y dy = \int_0^y \frac{1}{\sqrt{c_3}} \theta_y \sqrt{c_3} dy$$

then

$$(\theta - \theta_0)^2 \leqslant \int_0^l \frac{dy}{c_3} \cdot \int_0^l c_3 \theta_y^2 dy$$

But one can always assume that

$$\int_0^l \theta dy = 0$$

and thus the foregoing inequality leads to

$$\int_0^l c_3 \theta_y^2 dy \geqslant \|\theta\|^2 + l\theta_0^2 \left[l \left(\int_0^l \frac{dy}{c_3} \right)^{-1} \right]$$

$$\Pi^* \geqslant \|\theta\|^2 + l\theta_0^2 \left[l \left(\int_0^l \frac{dy}{c_3} \right)^{-1} \right] - \delta \|\theta\|^2$$

From this it follows that for positive definiteness of the functional Π^*, it is sufficient that function c_3 satisfies the inequality

$$\frac{1}{l} \left(\int_0^l \frac{dy}{c_3(y)} \right)^{-1} - \delta > 0 \tag{3.21}$$

This condition imposes a lower bound on the rigidity c_3. It will be necessarily satisfied if it is required that function c_3 satisfies the inequality

$$c_3(y) > \eta \tag{3.22}$$

where η is a positive number chosen in a suitable manner.

A condition of the form (3.22) can also be obtained in the case of arbitraty small vibrations. The following statement can be proved.

For positive definiteness of the functional Π^* it is sufficient that the functions C_i satisfy the inequalities

$$\min C_i > \eta_i, \qquad y \in [0l] \qquad (i = 1, 2, 3) \tag{3.23}$$

where η_i are some completely determined positive numbers. They depend on the character of the external forces and on the geometry of the liquid cavities and the density of the liquid.

3. Proof of the existence of principal modes. The problem of determining natural vibrations reduces to the study of the spectrum of the operator equation

$$\mathrm{L}x = \frac{1}{\omega}\mathrm{M}x$$

where the L and M are self-conjugate operators and M is positive definite. The L operator is not completely continuous. This circumstance requires some additional calculations.

We perform the substitution (3.14) in the expressions for T and Π and set up new equations of motion. In the simplest case of purely flexural vibrations this system becomes

$$AX_{tt} + \rho \int\limits_{l-d} \gamma_1 \mathrm{H}_1 \gamma_1 X_{tt} dl + \rho \int\limits_{l-d} \gamma_1 \mathrm{H}^* v_{tt} dl + (c_1 X_{yy})_{yy} + \beta_1{}^* X = 0$$

$$\mathrm{H}_1 \gamma_1 X_{tt} + \mathrm{H}^* v_{tt} + \rho g v = 0 \qquad (3.24)$$

Here

$$\mathrm{H}_1 = \mathrm{H} - \mathrm{H}^*, \qquad \mathrm{H}^* u = \int\limits_S H(P, Q)\, u(Q)\, ds_Q, \qquad \beta_1{}^* = \beta_1 - \rho g \int\limits_d \gamma_1{}^2 dl$$

Let us recall that $H(P, Q)$ is the Green function of the Neumann problem for the region τ. The outstanding feature of the transformed system (3.24) in comparison with the system (3.3) to (3.7) is the fact that its first three equations do not contain terms with v in them (the first equation contains only derivatives of that function with respect to t).

Introduce the function space $E^* = E_1 + E_2 + E_3 +$ and let $w^* \;\; E^*$.

Then the system of equations of motion in the general case can be written as follows:

$$Bw_{tt}{}^* + Dv_{tt} + Nw^* = 0, \qquad \rho H_1 \Gamma_1 w_{tt}{}^* + \rho H^* v_{tt} + \rho g v = 0 \qquad (3.25)$$

Here

$$B = (L_{ij}); \qquad \mathrm{D}v = \rho \int\limits_{l-d} \Gamma \mathrm{H}^* v dl, \quad \Gamma = \begin{Vmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{Vmatrix}, \qquad \Gamma_1 = \| \gamma_1, \gamma_2, \gamma_3 \|$$

The operators L^*_{ij} $(i = 1, 2, 3)$ differ from the operators L_{ij} by the fact that under the integral sign the operator H is replaced by H_1 and the integration extends over the interval $l - d$.

The symmetrical operator N is determined by the matrix $\| \beta_{ij} \|$, where the numbers β_{ij} $(i \neq j)$ are given by the formulas

$$\beta_{ij} = -\rho g \int\limits_d \gamma_i \gamma_j dl$$

It can be easily verified that the operator

$$\Lambda = \left\| \begin{matrix} B & D \\ \rho H_1 \Gamma_1 & H^* \end{matrix} \right\|$$

just as the operator L, is self-conjugate. In order to be convinced of this it is sufficient to check the validity of the equations

$$(\Lambda w, w_1) = (L\,x, x_1), \qquad (w, \Lambda\, w_1) = (x, L\, x_1), \qquad (w \in E) \qquad \left(w = \left\| \begin{matrix} w^* \\ v \end{matrix} \right\|\right)$$

The elements w and x are related by the equation (3.14).

The operator N is unbounded and positive definite. Its inverse N^{-1} is a completely continuous integral operator. The structure of this operator can best be seen by considering a particular case. For instance, if one studies purely torsional vibrations then the equation $Nu = f$ becomes

$$-(C_3\theta_y)_y + \beta\theta = f$$

and consequently

$$\theta(P) = \int_0^l G(P, Q) f(Q)\, dQ$$

where G is the Green function taking into account the corresponding boundary conditions and having a weak singularity.

To find periodic solutions let

$$w^* = w\cos\omega_n t, \qquad v = y\cos\omega_n t \tag{3.26}$$

System (3.25) becomes then

$$Bw + Dy = \frac{1}{\omega_n^2} Nw, \qquad \rho H_1\Gamma_1 w + \rho H^* y = \frac{1}{\omega^2}\rho g y \tag{3.27}$$

Here we change the variables $\phi = N^{1/2}w$ and $\phi = \sqrt{\rho}\, g\, y$. We obtain

$$\begin{gathered} N^{-1/2}BN^{-1/2}\varphi + N^{-1/2}D\frac{\psi}{\sqrt{\rho g}} = \frac{1}{\omega^2}\varphi \\ \frac{\rho}{\sqrt{\rho g}} H_1\Gamma_1 N^{-1/2}\varphi + \frac{\rho}{\sqrt{\rho g}} H^* \frac{\psi}{\sqrt{\rho g}} = \frac{1}{\omega^2}\psi \end{gathered} \tag{3.28}$$

Since the operator B is bounded, $N^{-1/2}BN^{-1/2}$ is completely continuous (inasmuch as N^{-1} is completely continuous). Operators D, H_1 and H^* are completely continuous as integral operators with weak singularities. Consequently, the operator that determines the left-hand side of the system (3.28) is completely continuous. Similarly it is easily verified that the operator

$$R = \left\| \begin{matrix} N^{-1/2}BN^{-1/2} & N^{-1/2}D\,\frac{1}{\sqrt{\rho g}} \\ \frac{\rho}{\sqrt{\rho g}}\,H_1\Gamma_1 N^{-1/2} & \frac{\rho}{\sqrt{\rho g}}\,H^*\,\frac{1}{\sqrt{\rho g}} \end{matrix} \right\|$$

is self-conjugate.

Thus we have arrived at the characteristic value problem

$$Rf = \lambda f$$

for the completely continuous self-conjugate operator R.

On the basis of known theorems of linear analysis we can establish the validity of the following fundamental theorem.

Theorem. If the quadratic form Π is positive definite then the system (3.3) to (3.7) has a periodic solution (principal modes of vibration) of the form (3.26), where ω_n are positive numbers (natural frequencies), forming a sequence such that $\omega_n \to \infty$ as $n \to \infty$.

This theorem contains Lagrange's Theorem on the minimum of potential energy as a special case of the given problem. The proved theorem indicates that if the potential energy of the beam-fluid system in the equilibrium state has a minimum, then this equilibrium state is stable in the sense that all principal oscillations of the system are bounded.

4. *Proof of completeness.* The system of principal modes of vibration is complete in E. In order to show this we shall write the fundamental operational equation in the form

$$M^{-1}Lx = \lambda x$$

Introduce the Friedrichs norm $(x, y)_F = (Mx, y)$. For the proof of completeness in the Friedrichs norm it is sufficient to show that from the condition $M^{-1}Lh = 0$ it follows immediately that $h \equiv 0$. Consider

$$(M^{-1}L\,h\cdot h)_F = (Lh\cdot h)$$

The scalar product on the right is the kinetic energy. Consequently, if $h \neq 0$ then this expression cannot be equal to zero. Consequently also $M^{-1}Lh \neq 0$.

The completeness of the system of principal modes allows to find the solution of the Cauchy problem in the form of a series composed of principal modes.

This completes the proof of the existence of a complete set of principal modes of the system studied. At the same time the proved statements substantiate the use of the Ritz method, whose scheme was discussed in detail in the first Section of this paper.

A short summary of several results of this article were published in the *Dokl. Akad. Nauk SSSR* (see [5]).

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